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ON THE ANALYTICAL DETERMINATION OF RELAXATION MODULUS OF VISCOELASTIC MATERIALS BY PRONY'S INTERPOLATION METHOD

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16. ABSTRACT

A computer implementation to Prony's curve fitting by exponential functions is presented. The method, although more than one hundred years old, has not been utilized to its fullest capabilities due to the restriction that the time range must be given in equal increments in order to obtain the best curve fit for a given set of data.

The procedure used in this paper utilizes the 3-dimensional capabilities of the Interactive Graphics Design System (I.G.D.S.) in order to obtain the equal time increments. The resultant information is then input into a computer program that solves directly for the exponential constants that yield the best curve fit. Once the exponential constants are known, a simple least squares solution can be applied to obtain the final form of the equation.

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ON THE ANALYTICAL DETERMINATION OF RELAXATION MODULUS OF VISCOELASTIC MATERIALS BY PRONY'S INTERPOLATION METHOD

INTRODUCTION

When analyzing materials with time dependent mechanical properties, it is of vital importance that the characterization of such properties be accurately defined. For viscoelastic materials, properties such as Relaxation Modulus, Creep Compliance, and time-dependent Poisson's Ratio are of utmost importance for numerical and closed-form solutions to various problems.

A popular method of obtaining analytical expressions for these properties consists of obtaining discrete values as a function of the logarithm of time and curve fitting the data to an appropriate expression. Due to the decaying nature of such properties as Relaxation Modulus and Poisson's Ratio, they are conveniently represented as a series of exponential functions. A widely used form of these functions is the so-called Prony series (due to Gaspard Francois Clair Marie Riche de Prony, 1755-1839), which can be expressed as,

$$\bar{f}(t) = A + \sum_{i=1}^{n} B_i \, e^{\gamma_i t} \quad .$$

When using expressions such as this for curve fitting it is noted that there are too many unknowns for the amount of equations available for simultaneous solving. This generally leads to a trial-and-error approach where values of γ_i are assumed, and an expression for the time-dependent variable is obtained. This process is repeated until a satisfactory curve fit has been obtained.

The intent of this paper is to demonstrate that with the use of the Intergraph Interactive Graphics Design System (I.G.D.S.) and its three-dimensional sculptured surfaces capability the true Prony method can be efficiently used, and the trial and error approach totally eliminated. This is accomplished by curve fitting the test data to a third-degree, B-Spline interpolation which can be done quickly and with double precision accuracy on the I.G.D.S. The resultant curve can then be divided into the desired number of equally spaced increments of time for the application of Prony's method. It must be stated at this time that equal time increments are necessary in order to apply the method properly. It will also be shown that the accuracy of the results is not compromised by the introduction of the intermediate step of the B-spline interpolation.

The method is applied in this paper to a solid rocket propellant (TP-H1148) and a fluorocarbon elastomer commonly used as O-ring material (V-747-75).

The calculations are performed on a Univac 1108 computer and the graphics on a VAX 780 computer, both with double precision accuracy.

DESCRIPTION OF PRONY'S METHOD

This method is well documented [7,1,6] and will be explained here as a means of describing the Fortran program PRONY which will solve, without trial and error, for the exponents γ_i.

Let Prony's equation be of the form

$$\bar{f}_{(t)_k} = A + \sum_{i=1}^{m} B_i \, e^{\gamma_i t_k} \quad , \tag{1}$$

where $k = 0,1,2,\ldots n$; $i = 1,2,3,\ldots m$; and $n \geq 2m - 1$. We first intend to obtain the value of γ_i and then the values of A and B_i. We must remember at this time that the successive values of time (abscissa) must form an arithmetic progression, that is $t_k = t_o + kw$, where w is the difference between any two successive values of time. Equation (1) can now be expressed as,

$$\bar{f}_{(t)_k} = A + \sum_{i=1}^{m} B_i \, e^{\gamma_i (t_o + kw)} \quad . \tag{2}$$

It is convenient to use the following abbreviations to simplify equation (2),

$$V_i = e^{\gamma_i w} \tag{3}$$

$$C_i = B_i \, e^{\gamma_i t_o} \quad , \tag{4}$$

therefore, equation (1) becomes the set of equations

$$\bar{f}_{(t)_k} = A + \sum_{i=1}^{m} C_i \, V_i^k \quad . \tag{5}$$

Willers [7] explains how the process of obtaining the γ_i can be simplified by forming the difference between two successive ordinates in order to conveniently eliminate A from the approximating equations. As an example, take 4 equidistant measurements and 3 exponential terms or $k = 0,1,2,3$ and $i = 1,2,3$; this will lead to the following set of equations (5),

$$\bar{f}_{(t)_0} = A + C_1 + C_2 + C_3 \tag{6a}$$

$$\bar{f}_{(t)_1} = A + C_1 V_1 + C_2 V_2 + C_3 V_3 \tag{6b}$$

$$\bar{f}_{(t)_2} = A + C_1 V_1{}^2 + C_2 V_2{}^2 + C_3 V_3{}^2 \tag{6c}$$

$$\bar{f}_{(t)_3} = A + C_1 V_1{}^3 + C_2 V_2{}^3 + C_3 V_3{}^3 \quad , \tag{6d}$$

the differences can be expressed as,

$$\bar{\Delta}_{1-0} = C_1(V_1-1) + C_2(V_2-1) + C_3(V_3-1) = \bar{f}_{(t)_1} - \bar{f}_{(t)_0} \tag{7a}$$

$$\bar{\Delta}_{2-1} = C_1(V_1-1)V_1 + C_2(V_2-1)V_2 + C_3(V_3-1)V_3 = \bar{f}_{(t)_2} - \bar{f}_{(t)_1} \tag{7b}$$

$$\bar{\Delta}_{3-2} = C_1(V_1-1)V_1{}^2 + C_2(V_2-1)V_2{}^2 + C_3(V_3-1)V_3{}^2 = \bar{f}_{(t)_3} - \bar{f}_{(t)_2} \tag{7c}$$

$$\bar{\Delta}_{4-3} = C_1(V_1-1)V_1{}^3 + C_2(V_2-1)V_2{}^3 + C_3(V_3-1)V_3{}^3 = \bar{f}_{(t)_4} - \bar{f}_{(t)_3} \quad . \tag{7d}$$

If we let the values of V_i be the roots of the equation

$$V^m + S_1 V^{m-1} + S_2 V^{m-2} + \ldots + S_{m-1} V + S_m = 0 \tag{8}$$

then we can solve for the values of γ_i from equations (3) or

$$\gamma_i = \frac{1}{w} \log_e V_i \quad . \tag{9}$$

When trying to interpolate with three or more exponential terms [$m \geq 3$ in equation (1)] there is the possibility of encountering complex roots V_i. This by no means precludes successful curve fitting. It does, however, change the final form of equation (1) thus forcing the analyst to utilize an expression with a combination of exponential and harmonic terms [1,7]. On the other hand, if the value of V_i is a negative

real value then equation (9) becomes unsolvable due to the fact that the natural logarithm of a negative number is undefined. Fortunately, it has been found that impressive accuracy can be obtained, for the materials in question, by using 2 exponential terms in equation (1), precluding the necessity for value of m greater than 2. Our problem now becomes that of obtaining the values of the S_i in equation (8). Equations (7) can exist simultaneously if we write

$$\overline{\Delta}_{1-0} S_3 + \overline{\Delta}_{2-1} S_2 + \overline{\Delta}_{3-2} S_1 + \overline{\Delta}_{4-3} = 0 \quad . \tag{10}$$

If we have a total of $n+1$ data points and m exponential terms in the Prony series, we can obtain a total of $n-m+1$ equations of the form (10) or

$$\overline{\Delta}_{1-0} S_m + \overline{\Delta}_{2-1} S_{m-1} + \cdots + \overline{\Delta}_{(m+1)-m} = 0$$

$$\overline{\Delta}_{2-1} S_m + \overline{\Delta}_{3-2} S_{m-1} + \cdots + \overline{\Delta}_{(m+2)-(m+1)} = 0$$

$$\overline{\Delta}_{3-2} S_m + \overline{\Delta}_{4-3} S_{m-1} + \cdots + \overline{\Delta}_{(m+3)-(m+2)} = 0 \tag{11}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots$$

$$\overline{\Delta}_{(n-m+1)-(n-m)} S_m + \overline{\Delta}_{(n-m+2)-(n-m+1)} S_{m-1} + \cdots + \overline{\Delta}_{n-(n-1)} = 0 \quad .$$

Equations (11) can readily be expressed in matrix form as

$$\begin{bmatrix} \overline{\Delta}_{1-0} & \overline{\Delta}_{2-1} & \cdots & \overline{\Delta}_{(m+1)-m} \\ \overline{\Delta}_{2-1} & \overline{\Delta}_{3-2} & \cdots & \overline{\Delta}_{(m+2)-(m+1)} \\ \vdots & \vdots & & \\ \overline{\Delta}_{(n-m+1)-(n-m)} & \overline{\Delta}_{(n-m+2)-(n-m+1)} & \cdots & \overline{\Delta}_{n-(n-1)} \end{bmatrix} \begin{Bmatrix} S_m \\ S_{m-1} \\ \vdots \\ 1 \end{Bmatrix} = 0 \tag{12}$$

or simply

$$[\overline{\delta}] \{S\} = 0 \tag{13a}$$

where the $\overline{\delta}$ matrix contains the $\overline{\Delta}_{k-(k-1)}$ coefficients and, in this case, $k = 1,2,3 \ldots n$. We can express (13a) in tensor notation as

$$\overline{\delta}_{qr} S_r = 0 \tag{13b}$$

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where

$$q = 1, 2, 3 \ldots n-m+1$$

$$r = 1, 2, 3 \ldots m+1 \quad ,$$

in this manner we have $\bar{\delta}_{11} = \bar{\Delta}_{1-0}$, $\bar{\delta}_{22} = \bar{\Delta}_{3-2}$, etc. From equations (13) we can form a set of normal equations [1,7] by the method of least squares which written in matrix form are

$$[\alpha] \{S\} = \{\varepsilon\} \quad . \tag{14a}$$

In the tensor notation we have

$$\alpha_{j\ell} S_\ell = \varepsilon_j \tag{14b}$$

where

$$j, \ell = 1, 2, 3, \ldots m$$

and

$$\alpha_{j\ell} = \sum_{q=1}^{n-m+1} \bar{\delta}_{qj} \bar{\delta}_{q\ell} \tag{15}$$

$$\varepsilon_j = - \sum_{q=1}^{n-m+1} \bar{\delta}_{qj} \bar{\delta}_{qp} \tag{16}$$

where $p = m+1$.

Once the α_{ij} and ε_j are obtained, equation (14a) can be solved for $\{S\}$ by premultiplying both sides of the equation by the inverse of $[\alpha]$ or

$$\{S\} = [\alpha]^{-1} \{\varepsilon\} \quad . \tag{17}$$

Equation (8) can now be solved for the V_i and from equation (9) the exponents γ_i can be obtained.

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COMPUTER IMPLEMENTATION

Table 1 is the listing of the FORTRAN program PRONY implementing the Prony method. The program reads the x_i and y_i, in this case time and Relaxation Modulus, and forms the α matrix. It is then inverted by means of the Gauss-Jordan reduction method and the values of S are obtained as in equation (17).

Once the S_i are obtained, then the γ_i can be obtained. It is then a simple step more to solve for the constants A and B_i by means of the least-squares method.

TABLE 1. FORTRAN PROGRAM PRONY

```
PEDROBIN197*ME799(1).PRONY
    1            PARAMETER N=32,M=2,NM1=N-1,MP1=M+1,M2=2*M
    2            PARAMETER NM1M=NM1-M,M2P2=2*M+2,M4P4=4*M+4
    3            DIMENSION JC(12)
    4            DOUBLE PRECISION ROOT(M2P2),AA(M2P2),BB(M2P2)
    5            DOUBLE PRECISION CC(M4P4),W,GAM,GAMMA
    6            DOUBLE PRECISION V(2),S(MP1)
    7            DOUBLE PRECISION E(N),T(N),DIFF(NM1),DELTA(NM1M,MP1)
    8            DOUBLE PRECISION ALPHA(M,MP1),EPS(M)
    9            DO 10 K=1,N
   10            READ(5,100)T(K),E(K)
   11            WRITE(10,100)T(K),E(K)
   12    100 FORMAT()
   13     10 CONTINUE
   14            DO 11 K=1,NM1
   15            KP1=K+1
   16            DIFF(K)=E(KP1)-E(K)
   17     11 CONTINUE
   18            DO 12 I=1,NM1M
   19            DO 12 J=1,MP1
   20            JJ=I+J-1
   21            DELTA(I,J)=DIFF(JJ)
   22     12 CONTINUE
   23            DO 14 I=1,M
   24            DO 14 J=1,M
   25            ALPHA(I,J)=0.0
   26            DO 14 K=1,NM1M
   27            ALPHA(I,J)=ALPHA(I,J)+DELTA(K,I)*DELTA(K,J)
   28     14 CONTINUE
   29            DO 15 I=1,M
   30            EPS(I)=0.0
   31            DO 15 J=1,NM1M
   32            EPS(I)=EPS(I)+DELTA(J,I)*DELTA(J,MP1)
   33     15 CONTINUE
   34            DO 16 I=1,M
   35            ALPHA(I,MP1)=-1.0*EPS(I)
   36     16 CONTINUE
   37            V(1)=4.0
   38            CALL DGJR(ALPHA,MP1,M,M,MP1,$300,JC,V)
   39    300 WRITE(6,301)JC(1)
   40    301 FORMAT(' SYSTEM SOLVED UP TO EQUATION NO. ',I3//)
   41            S(1)=1.0
   42            DO 17 I=1,M
   43            J=M-I+2
   44            S(J)=ALPHA(I,MP1)
   45     17 CONTINUE
   46            DO 18 I=1,MP1
   47            WRITE(6,400) I,S(I)
   48    400 FORMAT('   S(',I2,')=',E20.6)
   49     18 CONTINUE
   50            TOL=1.D-16
   51            CALL ROOTZ(S,M,ROOT,AA,BB,CC,TOL)
   52            WRITE(6,61)(ROOT(I),I=1,M2)
   53     61 FORMAT(20X,'ROOT',5X,2E30.8)
   54            W=T(3)-T(2)
   55            DO 19 I=1,M
   56            NN=2*I-1
   57            GAM=ROOT(NN)
   58            GAMMA=(DLOG(GAM))/W
   59            WRITE(6,700)GAMMA
   60    700 FORMAT(20X,'EXPONENT',E30.8)
   61     19 CONTINUE
   62            STOP
   63            END

@BRKPT PRINT$
```

NUMERICAL EXAMPLES

TP-H1148 SRM Propellant

Figure 1 shows a plot of the Relaxation Modulus versus log of time as obtained from a stress relaxation test [2]. As can be seen, the time range extends from 1.65958E-3 to 1.65958E+1 min. In order to properly apply this method, this data must be plotted in real time and then divided into equal time increments, w. This would be close to an impossible task unless performed on a graphics computer with the capabilities of I.G.D.S.
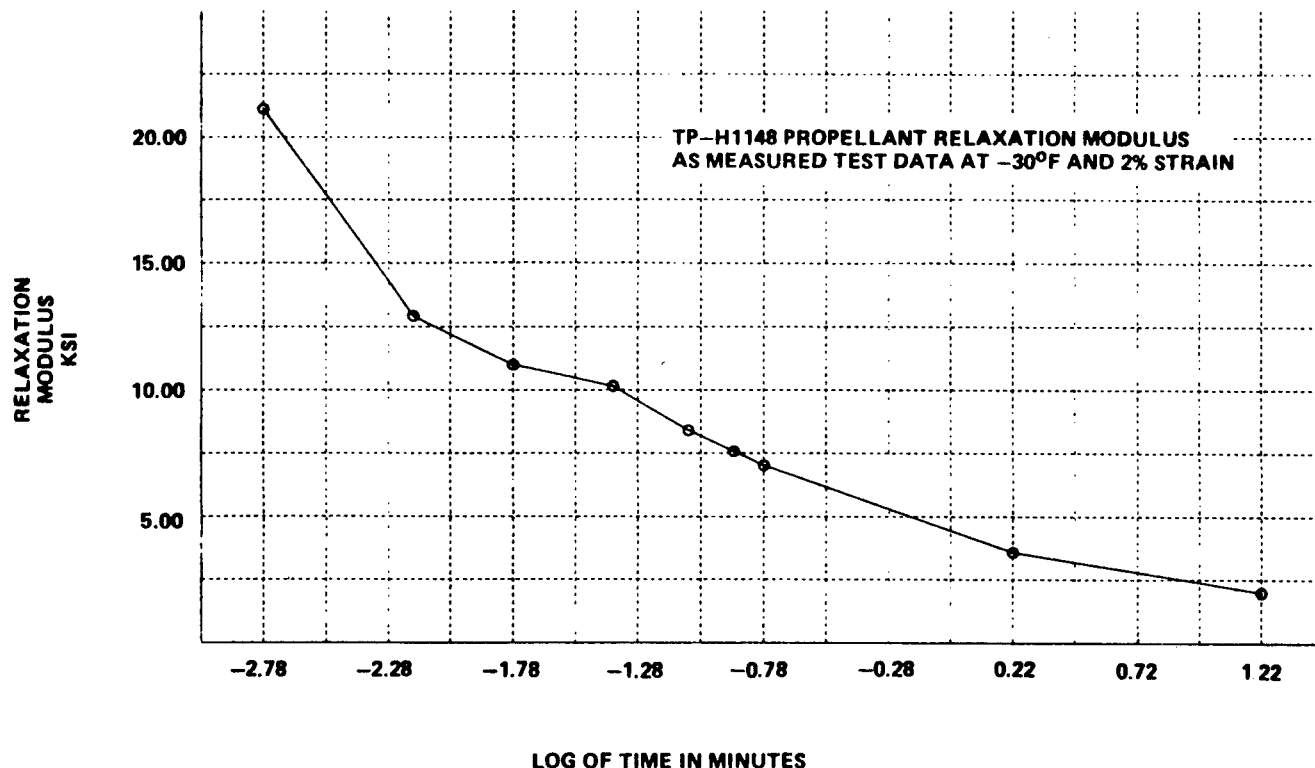

Figure 1. TP-H1148 Relaxation Modulus.

As can be seen in Figure 2a, we have plotted seven of the nine data points presented in Figure 1. This is done in order to show with clarity the shape of the curve, even though all nine points are used when performing the B-Spline interpolation shown in Figure 2b. As can be seen, the greatest relaxation occurs during the first 0.020 min of the test, and therefore we shall concentrate on this portion of the curve.

By "zooming-in" to the portion of interest, we can take increments of time of 0.001 min. Figure 3 shows the portion between 0.002 and 0.020 min of Figure 2b. The I.G.D.S. can accurately locate the intersection between the B-Spline curve and the time increments to yield values of Relaxation Modulus, E(t), at each intersection. These values can be read directly off the screen and tabulated as in Table 2. This is the basic information required in order to run PRONY.
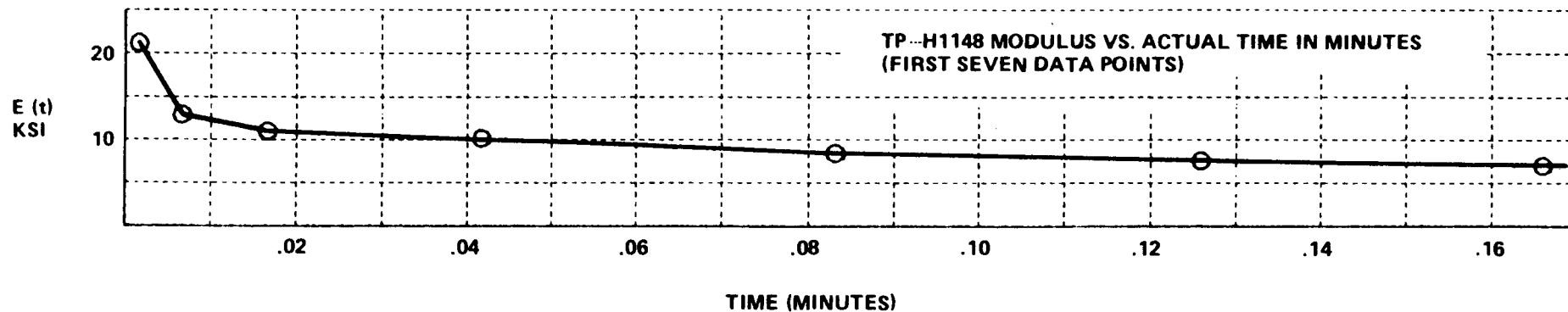
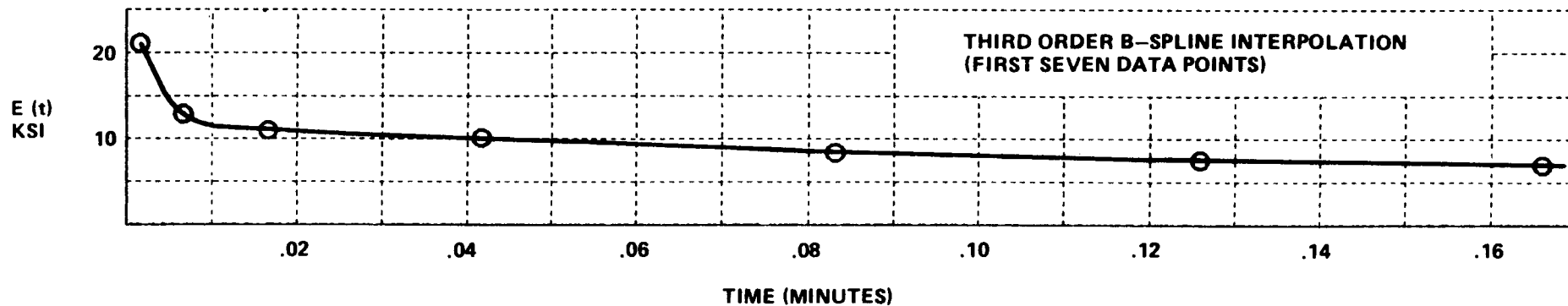Figure 2a. TP-H1148 modulus versus actual time.

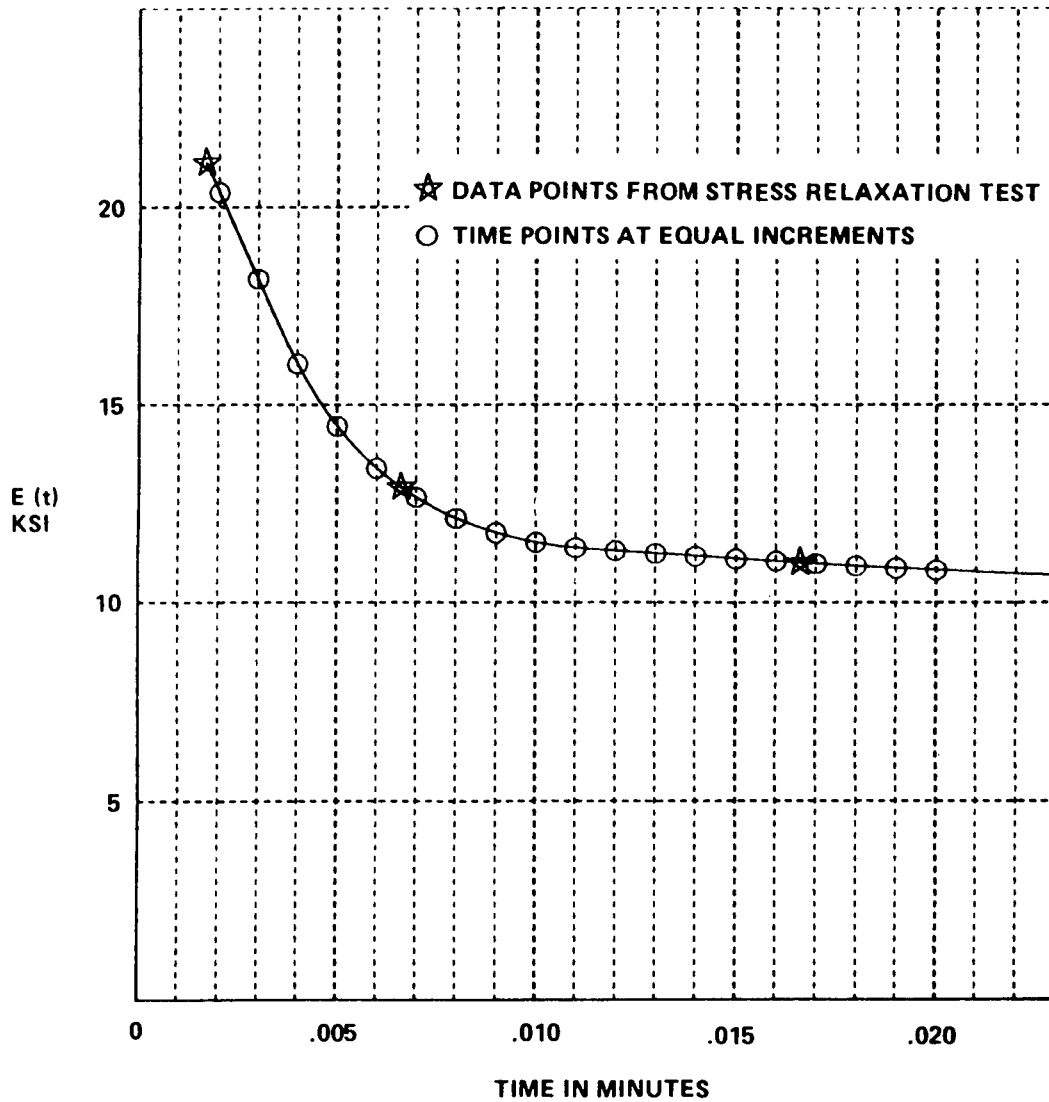Figure 2b. TP-H1148 B-Spline modulus versus actual time.

Figure 3. TP-H1148 modulus (first 0.020 min).

Equation (1) shows the general form of the Prony series used in our discussion. For this example let us take m = 2 since at m ≥ 3 we have encountered complex values of V_i. By inputing the data from Table 2 into our program we obtain the following form of equation (8):

$$V^2 - 0.822591 \, V + 0.095196 = 0 \qquad (18)$$

where,

$$S_1 = -0.822591$$

$$S_2 = 0.095196$$

The solution of equation (18) yields the following roots

TABLE 2. TP-H1148 PROPELLANT RELAXATION MODULUS

TIME (MINUTES)	E (t) (KSI)
.002	20.3704
.003	18.1835
.004	16.0257
.005	14.4590
.006	13.3989
.007	12.6508
.008	12.1197
.009	11.7508
.010	11.5090
.011	11.3706
.012	11.2952
.013	11.2244
.014	11.1563
.015	11.0909
.016	11.0282
.017	10.9683
.018	10.9112
.019	10.8569
.020	10.8056

$$V_1 = 0.6832668 \tag{19a}$$

$$V_2 = 0.1392342 \tag{19b}$$

Substitution of equations (19) into equations (9) yield the values of the required exponential term constants.

$$\gamma_1 = \frac{1}{0.001} \log_e 0.6832668 = -380.8698 \tag{20a}$$

$$\gamma_2 = \frac{1}{0.001} \log_e 0.1393242 = -1971.8239 \tag{20b}$$

We can now rewrite our Prony equation as

$$E(t) = A + B_1 e^{-380.8698\,t} + B_2 e^{-1971.8239\,t} \tag{21}$$

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where t = time in minutes, and A, B_1, and B_2 are constants to be determined by the method of least-squares [5]. These constants are,

$$A = 10973.3 \tag{22a}$$

$$B_1 = 23397.3 \tag{22b}$$

$$B_2 = -79056.7 \tag{22c}$$

It should be noted that equation (21) is valid only for the time range 0.002 to 0.020 min and that extrapolation beyond this range can lead to erroneous results. Figure 4 shows a plot of equation (21) as compared to the B-splined experimental data. As can be seen extremely good accuracy is obtained with only 2 terms.
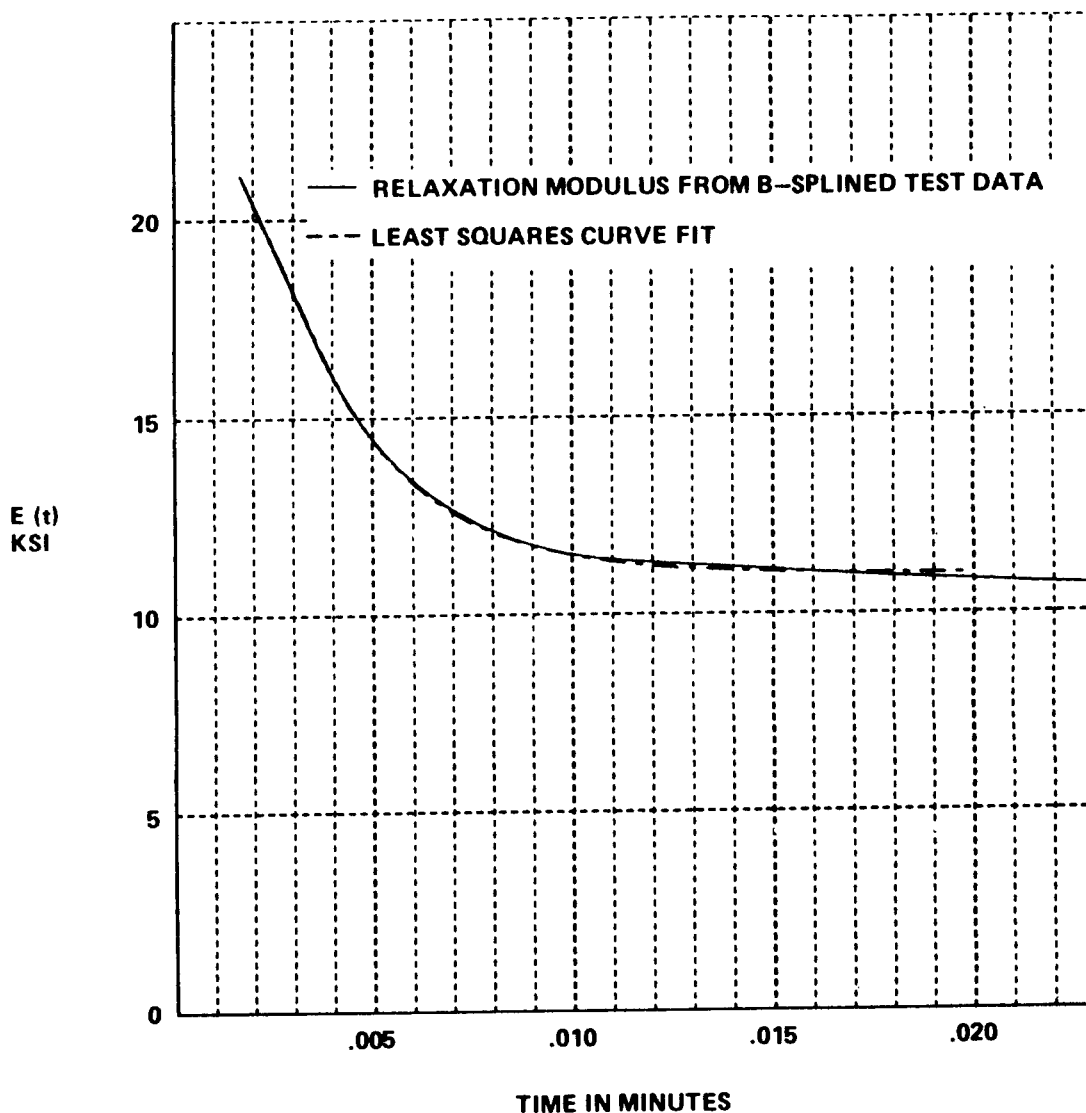

Figure 4. TP-H1148 curve fit (Prony's method).

Figure 5 shows a plot of the Relaxation Modulus versus log of time as obtained from a stress relaxation test [4]. Once again the time range extends from 1.65958E-3 to 1.65958E+1 min. The same procedure as for the TP-H1148 solid propellant is followed in order to obtain values of E(t) as a function of equal time increments w. Figures 6a and 6b are plots of actual data points and B-Spline interpolation, respectively.
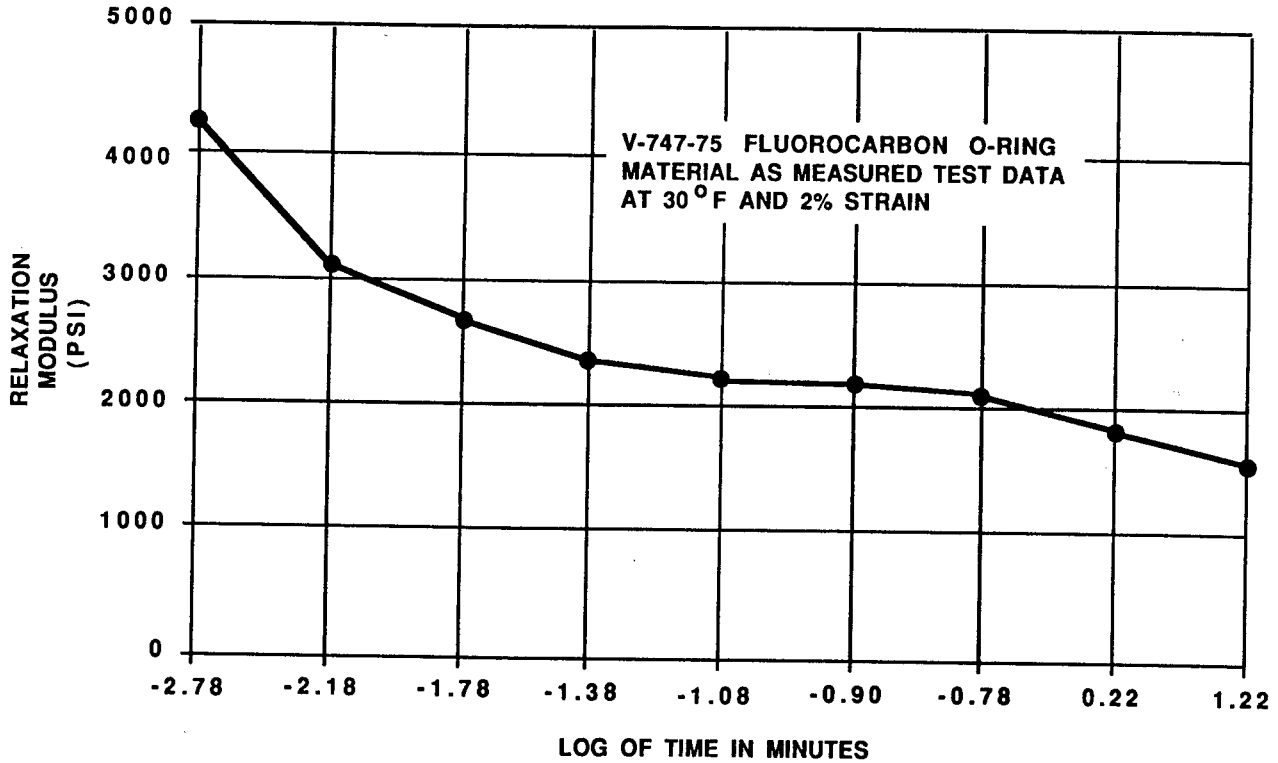

Figure 5. V-747-75 Relaxation Modulus.

For this example we shall take equal increments of time of 0.0025 min and analyze the portion of the curve that extends to 0.0800 min. Figure 7 shows the equal time increments, and Table 3 shows their respective values of Relaxation Modulus as obtained from the I.G.D.S. With this information we are again ready to run PRONY.

Again we have found that for $m \geq 3$ we encounter complex values of V_i; therefore, for $m = 2$ we obtain the following form of equation (8).

$$V^2 - 1.14458 \ V + 0.255224 = 0 \tag{23}$$

with

$$S_1 = -1.14458$$

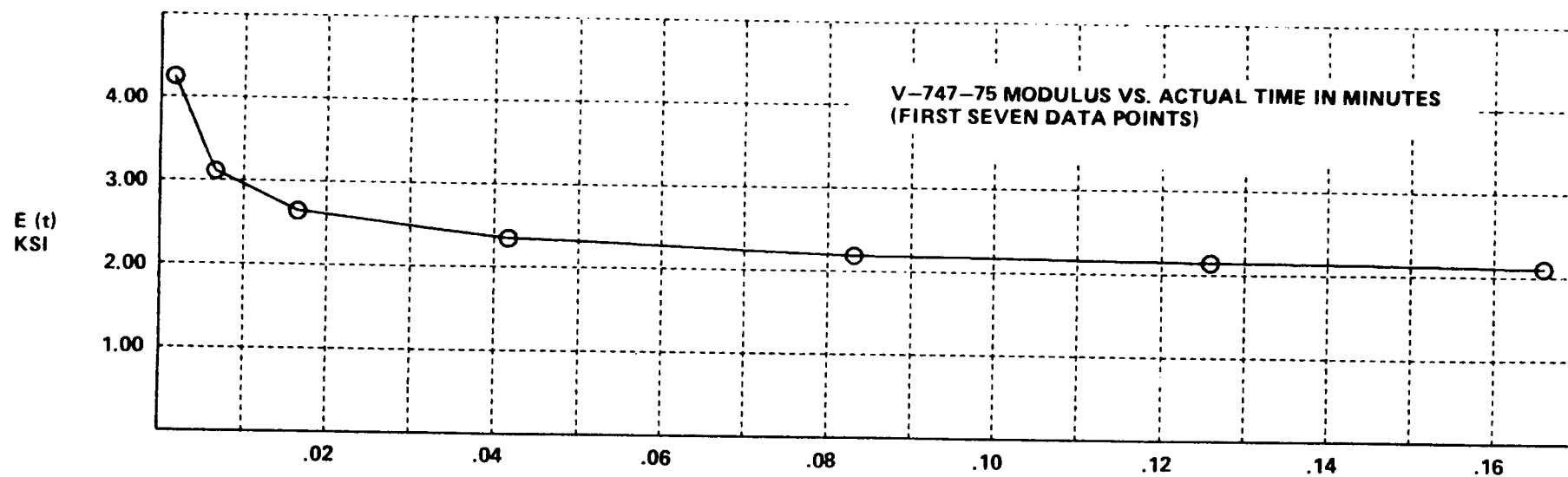$$S_2 = 0.255224 \quad .$$

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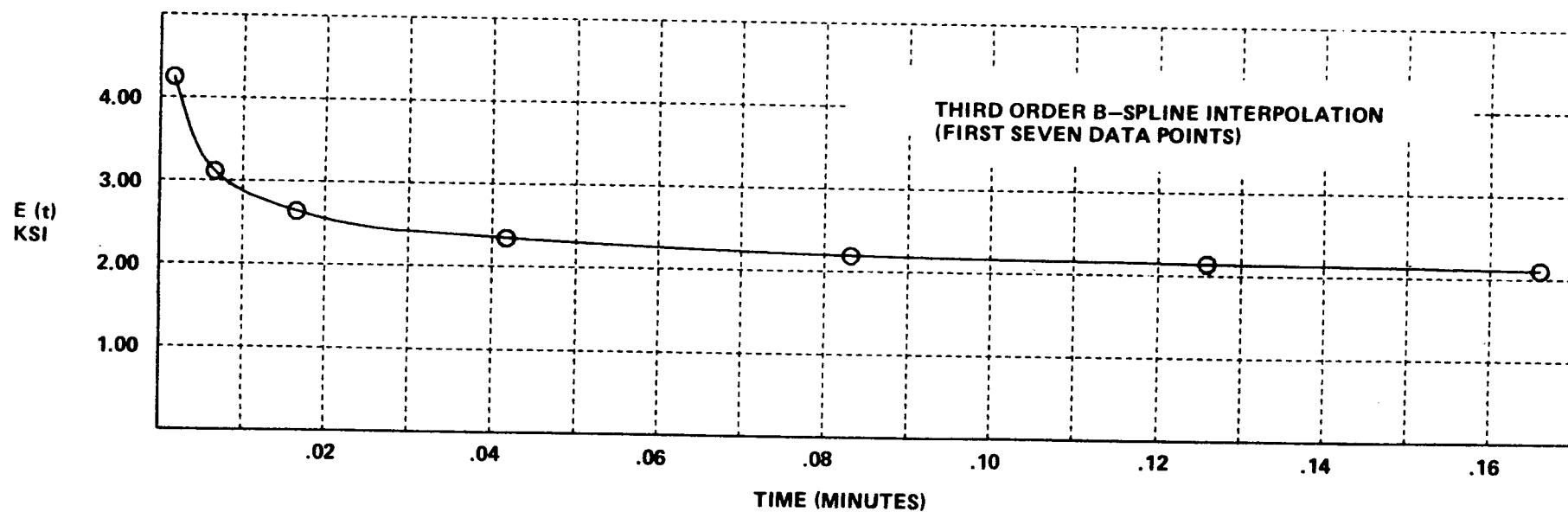
Figure 6a. V-747-75 Modulus versus actual time.

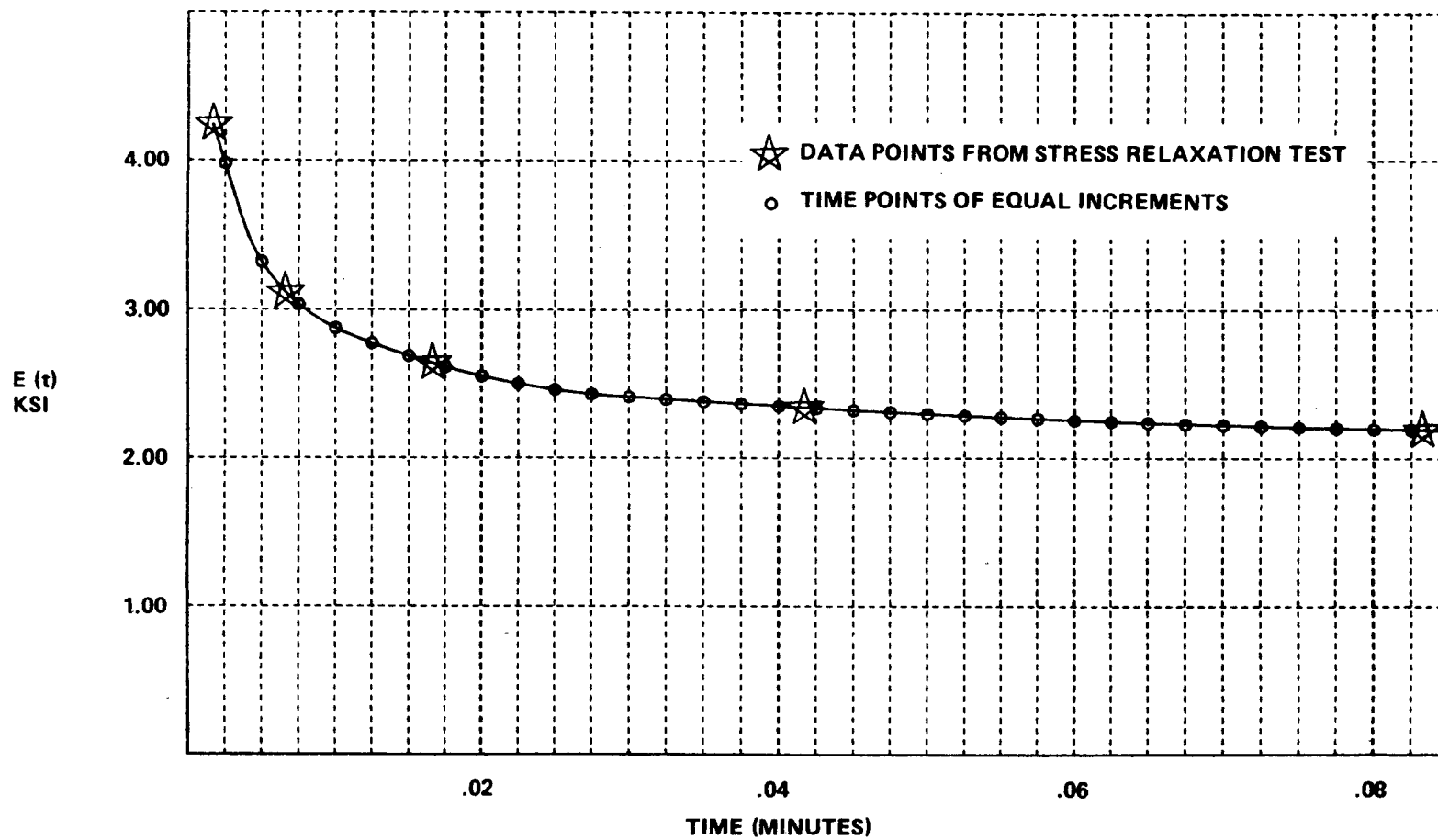Figure 6b. V-747-75 B-spline modulus versus actual time.

Figure 7.   V-747-75 Modulus (first 0.080 min).

TABLE 3. V-747-75 FLUOROCARBON O-RING MATERIAL RELAXATION MODULUS AS OBTAINED FROM FIGURE 7, FOR TIME BETWEEN 0.0025 AND 0.0800 MINUTES

TIME (MINUTES)	E (t) (KSI)
.0025	3977.9
.0050	3317.1
.0075	3032.1
.0100	2873.1
.0125	2772.4
.0150	2687.4
.0175	2613.8
.0200	2551.3
.0225	2499.9
.0250	2459.7
.0275	2430.5
.0300	2411.6
.0325	2395.4
.0350	2379.8
.0375	2364.8
.0400	2350.4
.0425	2336.6
.0450	2323.6
.0475	2310.7
.0500	2298.7
.0525	2287.2
.0550	2276.4
.0575	2266.1
.0600	2256.4
.0625	2247.3
.0650	2238.7
.0675	2230.6
.0700	2222.8
.0725	2215.5
.0750	2208.1
.0775	2202.1
.0800	2196.1

The roots of equation (23) are

$$V_1 = 0.84113 \tag{24a}$$

$$V_2 = 0.30345 \tag{24b}$$

The required exponential constants are obtained from equation (9) as

$$\gamma_1 = \frac{1}{w} \log_e V_1 = \frac{1}{0.0025} \log_e 0.84113 = -69.204 \tag{25a}$$

$$\gamma_2 = \frac{1}{w} \log_e V_2 = \frac{1}{0.0025} \log_e 0.30345 = -477.017 \tag{25b}$$

Our Prony equation for the V-747-75 material during the range $0.0025 \leq t \leq 0.0800$ is now

$$E_{(t)} = A + B_1 e^{-69.204t} + B_2 e^{-477.017t} \tag{26}$$

The constants A, B_1, and B_2 can be determined by the method of least squares [5],

$$A = 2239.88 \tag{27a}$$

$$B_1 = 1247.08 \tag{27b}$$

$$B_2 = 2253.44 \tag{27c}$$

Figure 8 shows a plot of equation (26) as compared to the B-Splined experimental data. Again a good correlation between the data and the curve fit is seen.

DISCUSSION OF RESULTS

Tables 4 and 5 show the comparison between the test data and the results from PRONY. We can see in both cases that there is very good accuracy between test points and curve fit points as long as we maintain ourselves between the time range under study or as determined by Tables 2 and 3. In both cases the value of E(t) departs from the actual values as soon as we leave the time range under study. At the beginning of the time range this problem can be overcome by extrapolating the the test data on to t = 0 [or t = 0.00001 min to avoid numerical problems when plotting Log(t)]. At the end of the time range the accuracy can be maintained by including as many time points as required for a specific analysis.
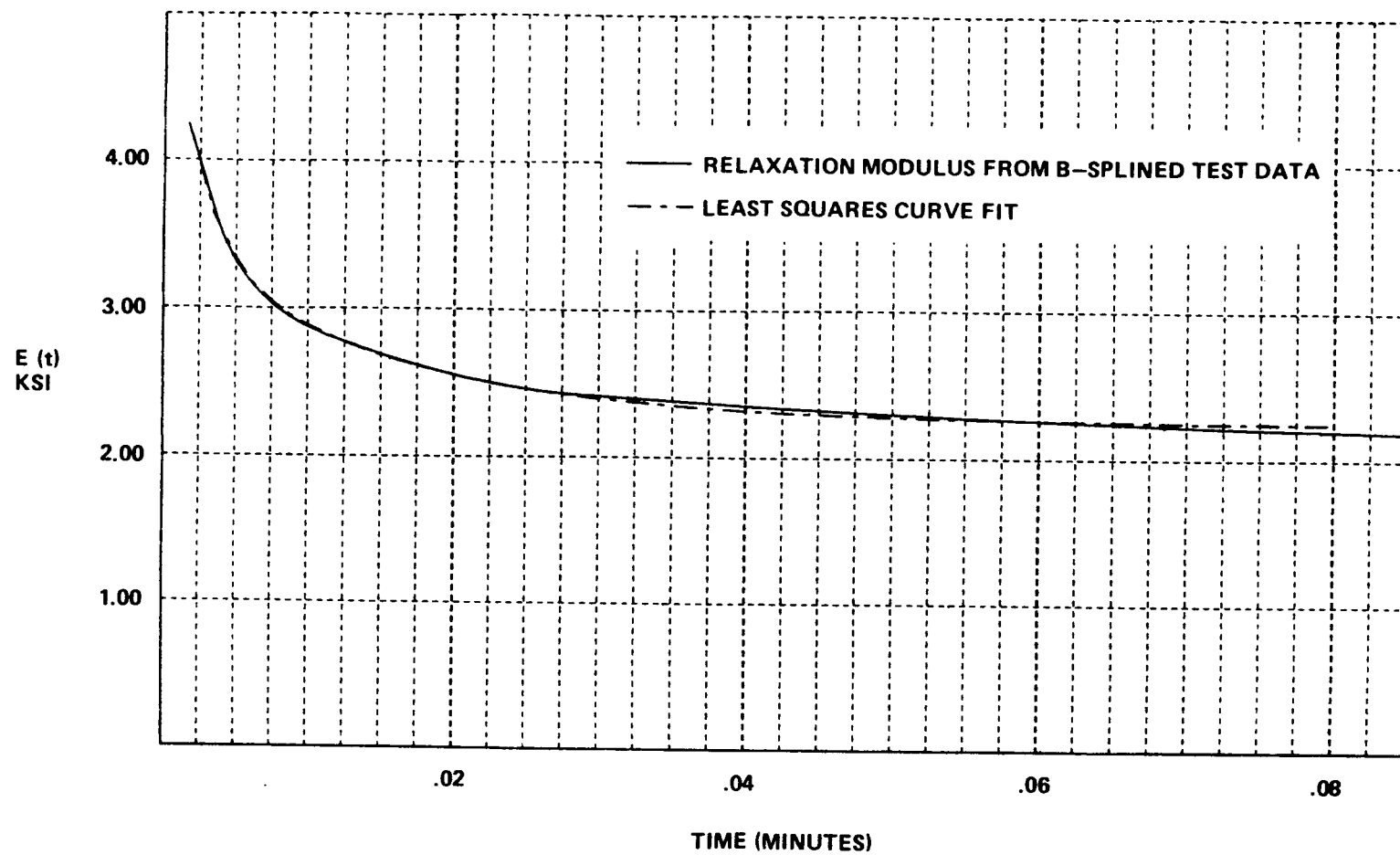
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Figure 8. V-747-75 curve fit (Prony's method).

TABLE 4. TP-H1148 RESULT COMPARISON CHART

Time (min)	Relaxation Modulus (ksi) $(0.002 \leq t \leq 0.020)$		
	Test Data	B-Spline	PRONY
0.00166	21.114	21.114	20.412
0.00661	12.913	12.913	12.861
0.01666	10.992	10.992	11.014
0.04168	10.144	10.144	10.973

TABLE 5. V-747-75 RESULT COMPARISON CHART

Time (min)	Relaxation Modulus (psi) $(0.0025 \leq t \leq 0.0800)$		
	Test Data	B-Spline	PRONY
0.00166	4247.00	4247.00	2636.05
0.00611	3113.00	3113.00	3179.15
0.01666	2639.00	2639.00	2634.39
0.04168	2341.00	2341.00	2309.58
0.08318	2189.00	2189.00	2243.82

Another point of interest is that if any of the roots of equation (8) are negative we will not be able to solve equation (9) for that particular γ_i since the natural logarithm of a negative number is undefined. A method to overcome this problem has not been addressed by any available source on the implementation of Prony's method [1,6,7]. It is the author's belief that due to the nature of equation (9) we must require that the values of V_i meet the following requirements:

1. All V_i shall be positive real values or complex conjugates.

2. If the data to be curve fitted is of a decaying nature as a function of time, the V_i must be less than 1.00 except for complex conjugate values.

3. If the data has increasing values of the ordinate with increasing time, then the V_i must be greater than zero, with at least one greater than 1, except for complex conjugate values.

CONCLUSIONS

A very good tool for the analytical determination of functions which can be represented by a series of exponential terms is available. The method, when utilized properly, can provide the analyst accurate expressions of the ordinate in question. Although the method leaves the unanswered question of what to do when V_i is a real negative number, the author feels that enough accuracy for materials such as elastomers and solid propellants can be obtained to perform an accurate stress and strain analysis of these materials under load.

APPENDIX A

CURVE FITTING PRONY'S EQUATION BY THE METHOD OF LEAST SQUARES

After obtaining the values of the γ_i exponents, Prony's equation (1) can be solved for the A and B_i constants from the data points by using the method of least squares [5,7].

The method, due to Legendre, is applied to Prony's equation as follows. Let Prony's equation be

$$\bar{f}(t) = A + \sum_{i=1}^{n} B_i e^{\gamma_i t} \quad . \qquad (A-1)$$

If we want to approximate a curve with f(t) as its discrete data points, we can express a deviation for the argument value of t as,

$$\delta_m = \bar{f}(t)_m - f(t)_m \qquad (A-2)$$

where m = 1,2,3,...,k and k is the total number of data points. The sum of the squares of all the deviations is expressed as,

$$D = \sum_{m=1}^{k} (\bar{f}(t)_m - f(t)_m)^2 = \sum_{m=1}^{k} \delta_m^2 \qquad (A-3)$$

If we substitute equation (A-1) into equation (A-3), we obtain,

$$D = \sum_{m=1}^{k} (A + \sum_{i=1}^{n} B_i e^{\gamma_i t_m} - f(t)_m)^2 \quad . \qquad (A-4)$$

The method of least squares requires that equation (A-4) be a minimum. In order for this to be true it is necessary that

$$\frac{\partial D}{\partial A} = \frac{\partial D}{\partial B_i} = 0 \quad . \qquad (A-5)$$

From equations (A-5) and (A-4) we can write,

$$\frac{1}{2} \frac{\partial D}{\partial A} = \sum_{m=1}^{k} (A + \sum_{i=1}^{n} B_i e^{\gamma_i t_m} - f(t)_m) \frac{\partial \bar{f}(t)_m}{\partial A} = 0 \qquad (A-6)$$

$$\frac{1}{2} \frac{\partial D}{\partial B_j} = \sum_{m=1}^{k} (A + \sum_{i=1}^{n} B_i e^{\gamma_i t_m} - f(t)_m) \frac{\partial \bar{f}(t)}{\partial B_j} = 0 \qquad (A-7)$$

where $j = 1, 2, 3 \ldots n$.

Equation (A-7) constitutes a set of n equations or one equation for each exponential term. These latter two equations can be rewritten as

$$A \sum_{m=1}^{k} \frac{\partial \bar{f}(t)_m}{\partial A} + \sum_{i=1}^{n} B_i \sum_{m=1}^{k} e^{\gamma_i t_m} \frac{\partial \bar{f}(t)_m}{\partial A} = \sum_{m=1}^{k} f(t)_m \frac{\partial \bar{f}(t)_m}{\partial A} \qquad (A-8)$$

$$A \sum_{m=1}^{k} \frac{\partial \bar{f}(t)_m}{\partial B_j} + \sum_{i=1}^{n} B_i \sum_{m=1}^{k} e^{\gamma_i t_m} \frac{\partial \bar{f}(t)_m}{\partial B_j} = \sum_{m=1}^{k} f(t)_m \frac{\partial \bar{f}(t)_m}{\partial B_j} \qquad (A-9)$$

For the purpose of writing a computer program that will solve for the constants A and B_j, equations (A-8) and (A-9) may be expressed in matrix form as follows,

$$[C] \{A\} = \{F\} \qquad (A-10)$$

where the vector $\{A\}$ holds the sought unknowns A and B_j and the coefficients of [C] and $\{F\}$ are defined as follows,

$$C_{11} = \sum_{m=1}^{k} \frac{\partial \bar{f}(t)_m}{\partial A} = k \qquad (A-11)$$

$$C_{1i} = \sum_{m=1}^{k} e^{\gamma_{i-1} t_m} \frac{\partial \bar{f}(t)_m}{\partial A} \qquad (A-12)$$

$$C_{i1} = \sum_{m=1}^{k} \frac{\partial \bar{f}(t)_m}{\partial B_i} \qquad (A-13)$$

$$C_{ji} = \sum_{m=1}^{k} e^{\gamma_{i-1}t_m} \frac{\partial \bar{f}(t)_m}{\partial B_{j-1}} \tag{A-14}$$

For equation (A-14) only we have

$$i = j = 2, \ 3, \ \ldots \ k+1 \tag{A-15a}$$

whereas for equations (A-12) and (A-13),

$$i = 1, \ 2, \ 3, \ \ldots \ k \qquad . \tag{A-15b}$$

The coefficients of the vector {F} are defined as,

$$F_1 = \sum_{m=1}^{k} f(t)_m \frac{\partial \bar{f}(t)_m}{\partial A} \tag{A-16}$$

$$F_i = \sum_{m=1}^{k} f(t)_m \frac{\partial \bar{f}(t)_m}{\partial B_{i-1}} \tag{A-17}$$

and in this case the i is defined as in equation (A-15a).

Once the C and F coefficients are properly identified, both sides of equation (A-10) can be premultiplied by the inverse of [C] to obtain the solution vector {A} or,

$$\{A\} = [C]^{-1} \{F\} \tag{A-18}$$

Table A-1 shows a printout of the program CFIT applying the method described above.

TABLE A-1. FORTRAN PROGRAM CFIT

```
PEDROBIN197*ME799(1).CFIT
    1              PARAMETER K=2,KP1=K+1,KP2=K+2
    2              PARAMETER M=33,VAL=100.,MM=100
    3              DIMENSION JC(12),E(M),T(M)
    4              DOUBLE PRECISION B(K),F(KP1),C(KP1,KP2),PRONY(MM)
    5              DOUBLE PRECISION V(2),MOD(MM)
    6              DOUBLE PRECISION TT(MM+1),R(K)
    7     C            INITIALIZE MATRIX AND VECTOR
    8              DO 10 N=1,KP1
    9              F(N)=0.0
   10              DO 10 L=1,KP2
   11              C(N,L)=0.0
   12         10 CONTINUE
   13              R(1)=118.159
   14              R(2)=661.228
   15     C        R(3)=415.608
   16     C        R(4)=500.
   17              DO 11 I=1,M
   18              READ(5,100)T(I),E(I)
   19        100 FORMAT()
   20         11 CONTINUE
   21              C(1,1)=1.0*M
   22              DO 20 N=2,KP1
   23              DO 20 I=1,M
   24              C(1,N)=C(1,N)+EXP(-R(N-1)*T(I))
   25              C(N,1)=C(N,1)+EXP(-R(N-1)*T(I))
   26         20 CONTINUE
   27              DO 30 J=2,KP1
   28              DO 30 N=2,KP1
   29              DO 30 I=1,M
   30              C(J,N)=C(J,N)+EXP(-R(N-1)*T(I))*EXP(-R(J-1)*T(I))
   31         30 CONTINUE
   32              DO 40 I=1,M
   33              F(1)=F(1)+E(I)
   34         40 CONTINUE
   35              DO 50 N=2,KP1
   36              DO 50 I=1,M
   37              F(N)=F(N)+E(I)*EXP(-R(N-1)*T(I))
   38         50 CONTINUE
   39              DO 60 N=1,KP1
   40              C(N,KP2)=F(N)
   41         60 CONTINUE
   42              V(1)=4.0
   43              CALL DGJR(C,KP2,KP1,KP1,KP2,$300,JC,V)
   44        300 WRITE(6,301)JC(1)
   45        301 FORMAT(' SYSTEM SOLVED UP TO EQUATION NO.',I3//)
   46              A=C(1,KP2)
   47              DO 70 N=2,KP1
   48              B(N-1)=C(N,KP2)
   49         70 CONTINUE
   50              WRITE(6,102)A,B(1),B(2),B(3)
   51        102 FORMAT(10X,4E15.6)
   52              TMAX=T(M)
   53              TMIN=T(1)
   54              DO 80 I=1,MM
   55              PRONY(I)=0.0
   56         80 CONTINUE
   57              XINC=(TMAX-TMIN)/VAL
   58              TT(1)=TMIN
   59              DO 90 I=1,MM
   60              DO 101 N=2,KP1
   61              PRONY(I)=PRONY(I)+B(N-1)*EXP(-R(N-1)*TT(I))
   62        101 CONTINUE
   63              MOD(I)=A+PRONY(I)
   64              TT(I+1)=TT(I)+XINC
   65         90 CONTINUE
   66              ARG=TMIN
   67              Y1=0.0
   68              Y2=0.0
   69              ARG1=0.0
   70              DO 110 I=1,MM
   71              WRITE(9)ARG1,Y1,Y2
   72              ARG1=ALOG10(ARG)
   73              CALL INTRP5(ARG,T,E,M,2,Y1,N)
   74              Y2=MOD(I)
   75              ARG=ARG+XINC
   76     C        WRITE(6,105)ARG,Y1,Y2
   77     C   105 FORMAT(10X,3E20.6)
   78        110 CONTINUE
   79              STOP
   80              END
```
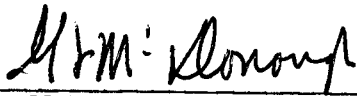
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ON THE ANALYTICAL DETERMINATION OF RELAXATION MODULUS OF
VISCOELASTIC MATERIALS BY PRONY'S INTERPOLATION METHOD

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The information in this report has been reviewed for technical content. Review
of any information concerning Department of Defense or nuclear energy activities or
programs has been made by the MSFC Security Classification Officer. This report,
in its entirety, has been determined to be unclassified.

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